Exam Calculus of Variations and Optimal Control 2015-16

Date	:	22-01-2015
Place	:	5161.0165
Time	:	14.00-17.00

The exam is OPEN BOOK; you can use all your books/papers/notes.
You are supposed to provide arguments to all your answers, and to explicitly refer to theorems/propositions whenever you use them.

1. A classical problem of maximizing the area enclosed by a curve of given length (Dido's problem) leads to maximizing the expression

   $$\int_0^1 x(t)\sqrt{1-\dot{x}^2(t)}\,dt,$$

   over all functions $x : [0,1] \to \mathbb{R}$ with $x(0) = x_0, x(1) = x_1$, where x_0, x_1 are fixed.

   (a) Derive the Euler-Lagrange equation for this problem.

   (b) Modify the problem by maximizing $\int_0^1 x(t)\sqrt{1-\dot{x}^2(t)}dt$ over all functions $x(\cdot)$ with $x(0) = x_0$ (no end condition). How does this affect the solution of the problem ?

   (c) The Euler-Lagrange equation determined in part (a) is quite complicated.
   Use the Beltrami-identity to prove that maximizing solutions $x(\cdot)$ are solutions of the differential equation

   $$\dot{x}^2 = cx^2 + 1$$

   for a certain constant c.

2. Consider the problem of minimizing the cost $x(1)$ for the system

   $$\dot{x} = xu, \quad u \in [-1,1],\ x(0) = x_0$$

   (a) Show by inspection that for fixed $x_0 > 0$ the optimal control is $u(t) = -1, t \in [0,1]$, and the minimal cost is $V(x_0) = e^{-1}x_0$.
   Show by inspection that for fixed $x_0 < 0$ the optimal control is $u(t) = 1, t \in [0,1]$, and the minimal cost is $V(x_0) = ex_0$.
   What are the minimal costs for $x_0 = 0$?
   Conclude that the value function for $t = 0$ is *not* differentiable at $x_0 = 0$.

   (b) Write out the equations of the Maximum principle for this problem. That is, determine the Hamiltonian H, and write out the differential equation (plus boundary conditions) for the co-state p, as well as the necessary conditions for the optimal control u.

   Show how you recover the same conclusions regarding the optimal control as in part (a). In particular, show that along the optimal trajectory the co-state is always positive.

3. Consider the scalar system

$$\dot{x} = u, \quad x, u \in \mathbb{R},\ x(0) = x_0,$$

and the cost criterion

$$\int_0^1 [\frac{1}{2}x^2(t) + \frac{1}{2}u^2(t) - \lambda x(t)u(t)]\, dt + \frac{1}{2}x^2(1)$$

   (a) Derive the Bellman equation.

   (b) Substitute in the Bellman equation the candidate value function V of the form

   $$V(x,t) = x^2 p(t)$$

   and derive an ordinary differential equation in the unknown function $p(t)$ with boundary condition. What is the optimal control in terms of $p(t)$ and $x(t)$?

   (c) Consider the special case $\lambda = 1$. What is the optimal control in this case, and how could you have seen this by inspection ?

4. Consider the nonlinear system

$$\begin{aligned} \dot{x}_1 &= x_1 + e^{x_2} - 2 + u \sin x_2 \\ \dot{x}_2 &= -\sin x_2 + (x_1 - 1)^2 + u \end{aligned}$$

   (a) Prove that for $u = 0$ the point $(\bar{x}_1, \bar{x}_2) = (1, 0)$ is an unstable equilibrium.

   (b) Can the equilibrium point $(\bar{x}_1, \bar{x}_2) = (1, 0)$ be made asymptotically stable by applying a feedback $u = \alpha(x)$? If so, determine such a feedback.

Distribution of points: Total 100; Free 10.

1. a: 5, b: 5, c: 10.
2. a: 10, b: 15.
3. a: 10, b: 10, c: 5.
4. a: 10, b: 10.